Lecture 8

Note Title 5/10/2009

In the last lecture we studied the geostrophic adjustment problem. Specifically we looked at how a step in the free surface of a shallow layer of water adjusts under the constraint of potential vorticity conservation.

With rotation as $t \to \infty$

$$\eta = \eta_0 \begin{cases} 1 - \exp(x/Lr) & x < 0 \\ \exp(-x/Lr) - 1 & x > 0 \end{cases}$$

Owing to conservation of PV, the free surface does not completely flatten out unlike the case of the non-rotating fluid.

Expressed in terms of the energetics of the system not all of the initial potential energy (PE) is converted to kinetic energy (KE), showing this quantitatively:

The gravitational potential energy of a body with mass m is

$$PE = mgz + \text{const}$$

Therefore for a fluid, the PE per unit volume is:

$$\frac{PE}{\text{volume}} = \rho g z$$

The total PE of the fluid is thus

$$PE = \iiint \rho g z \, dz \, dx \, dy$$

For the shallow water system, with uniform density $\rho = \rho_0$ the PE is thus:

$$PE = \iint \int_0^{H+\eta} \rho_0 g z \, dz \, dx \, dy$$

$$= \iint \frac{\rho_0 g}{2} z^2 \Big|_0^{H+\eta} dx \, dy$$

$$PE = \frac{\rho_0 g}{2} \iint (H^2 + H\eta + \eta^2) \, dx \, dy$$

Assuming that the average value of the free surface is zero, i.e.

$$\iint \eta \, dx \, dy = 0$$

Then the PE is (after removing the constant $\frac{\rho_0 g}{2}\iint H^2 dxdy$) is

$$\boxed{PE = \frac{\rho_0 g}{2}\iint \eta^2\,dx\,dy}$$

The total KE is simply:

$$\boxed{KE = \frac{1}{2}\rho_0\int_0^{H+\eta}\iint (u^2+v^2)\,dx\,dy\,dt}$$

Calculating the change in PE for the geostrophic adjustment problem:

$$\frac{\Delta PE}{\text{length}} = \frac{\rho_0 g}{2}\int_{-\infty}^{\infty}\left(\eta^2\Big|_{t\to\infty} - \eta^2\Big|_{t=0}\right)dx$$

$$\eta^2\Big|_{t\to\infty} - \eta^2\Big|_{t=0} = \eta_0^2\begin{cases}(1-e^{x/Lr})^2 - 1 = +e^{2x/Lr} - 2e^{x/Lr} & x<0\\ (e^{-x/Lr}-1)^2 - 1 = e^{-2x/Lr} - 2e^{-x/Lr} & x>0\end{cases}$$

$$\frac{\Delta PE}{\text{Length}} = \frac{\rho_0 g\eta_0^2}{2}\left[\int_{-\infty}^{0}\left(e^{2x/Lr} - 2e^{x/Lr}\right)dx + \int_0^{\infty}\left(e^{-2x/Lr} - 2e^{-x/Lr}\right)dx\right]$$

$$= \frac{\rho_0 g \eta_0^2}{2}\left[\frac{L_r}{2} - 2L_r + \frac{L_r}{2} - 2L_r\right]$$

$$\boxed{\frac{\Delta PE}{\text{length}} = -\frac{3}{2}\rho_0 g \eta_0^2}$$

Note that the change in PE is FINITE this in contrast to what would occur in a non rotating fluid.

What about the KE? As $t \to \infty$

$$v = -\frac{\eta_0 g}{f L_r} e^{-|x|/L_r}, \quad u = 0$$

$$\frac{KE}{\text{length}} = \frac{1}{2}\rho_0 \int_0^{H+\eta}\int (u^2+v^2)\,dx\,dz \approx \frac{1}{2}\rho_0 H \int (u^2+v^2)\,dx$$

$$\eta \ll H$$

$$\left.\frac{KE}{\text{length}}\right|_{t\to\infty} = \frac{1}{2}\rho_0 H \left(\frac{\eta_0 g}{f L_r}\right)^2 \int_{-\infty}^{\infty} e^{-2|x|/L_r}\,dx$$

$$= \frac{1}{2}\rho_0 g \eta_0^2 \left[\int_{-\infty}^{0} e^{+2x/L_r}\,dx + \int_0^{\infty} e^{-2x/L_r}\,dx\right]$$

$$\boxed{\left.\frac{KE}{\text{length}}\right|_{t\to\infty} = \frac{1}{2}\rho_0 g \eta_0^2}$$

We see that the decrease in PE is three times larger in magnitude than the increase in KE:

$$\left|\frac{\Delta PE}{\text{length}}\right| = 3\,\frac{\Delta KE}{\text{length}}$$

Where does the rest of this PE go?

$\Rightarrow$ Into TRANSIENT MOTIONS

→ show PPT of Gill's Figure

What are the properties of these transient, unbalanced motions for the shallow water system?

These transient motions are known as Poincaré waves whose motions are governed by the linearized shallow water equations:

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial h}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial h}{\partial y}$$

$$\frac{\partial h}{\partial t} = -H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

We wish to calculate the dispersion relation for these waves and thus look for plane wave solutions:

$$\begin{pmatrix} u \\ v \\ \eta \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \\ \eta_0 \end{pmatrix} e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

Substituting this form into the equations

$$-i\omega u_0 - f v_0 = -igk\eta_0 \qquad Ⓐ$$

$$-i\omega v_0 + f u_0 = -igl\eta_0 \qquad Ⓑ$$

$$-i\omega \eta_0 = -iH(ku_0 + lv_0) \qquad Ⓒ$$

Assuming that u_0, k, l are known
solve for v_0, η_0, ω

$$\eta_0 = \frac{H}{\omega}(ku_0 + lv_0) \qquad \text{From Ⓒ}$$

$$-\omega i v_0 + f u_0 = -ig\frac{lH}{\omega}(ku_0 + lv_0) \qquad \text{from Ⓑ}$$

$$-i\omega^2 v_0 + f\omega u_0 = -iglH(ku_0 + lv_0)$$

$$[-i\omega^2 + igl^2H]v_0 = -f\omega u_0 - iglHku_0$$

$$v_0 = \frac{(-f\omega - iglkH)}{-i\omega^2 + igl^2H} u_0$$

$$v_0 = \frac{(if\omega - gHlk)}{(gHl^2 - \omega^2)} u_0$$

$$\eta_0 = \frac{H}{\omega}\left[k + \frac{l(if\omega - gHlk)}{(gHl^2 - \omega^2)}\right] u_0$$

$$\eta_0 = \frac{H}{\omega(gHl^2-\omega^2)}\left[k(gHl^2-\omega^2)+l(if\omega - gHlk)\right]u_0$$

$$\eta_0 = \frac{H}{\omega(gHl^2-\omega^2)}\left[-\omega^2 k + ilf\omega\right]u_0$$

$$\eta_0 = \frac{H}{gHl^2-\omega^2}\left[ilf - \omega k\right]u_0$$

Thus the POLARIZATION RELATIONS, i.e. the equations relating the various quantities of the waves (u, v, η_0) are:

$$\boxed{\begin{aligned} v_0 &= \frac{(if\omega - gHlk)}{(gHl^2-\omega^2)}\,u_0 \\ \eta_0 &= H\,\frac{(ilf - \omega k)}{(gHl^2-\omega^2)}\,u_0 \end{aligned}}$$

We can substitute these relations into equation Ⓐ to solve for the frequency ω:

$$-i\omega u_0 + \frac{(f^2\omega + igflkH)}{igl^2H - i\omega^2}\,u_0 = -ig\frac{k}{\omega}Hku_0$$
$$-i\frac{gkH}{\omega}l\frac{(-f\omega - iglkH)}{(igl^2H - i\omega^2)}u_0$$

Multiply through by $\dfrac{\omega(igl^2H - i\omega^2)}{u_0}$

$$-i\omega^2(igl^2H - i\omega^2) + \omega(f^2\omega + i\cancel{gfl}kH) =$$

$$-igk^2H(i\cancel{gl^2}H - i\omega^2) + igkHl(\cancel{f}\omega + igl\cancel{k}H)$$

$$\omega^2 f^2 - \omega^4 + \omega^2 gHl^2 + \omega^2 gk^2H = 0$$

$$\omega^2\left[\omega^2 - f^2 - gH(l^2 + k^2)\right] = 0$$

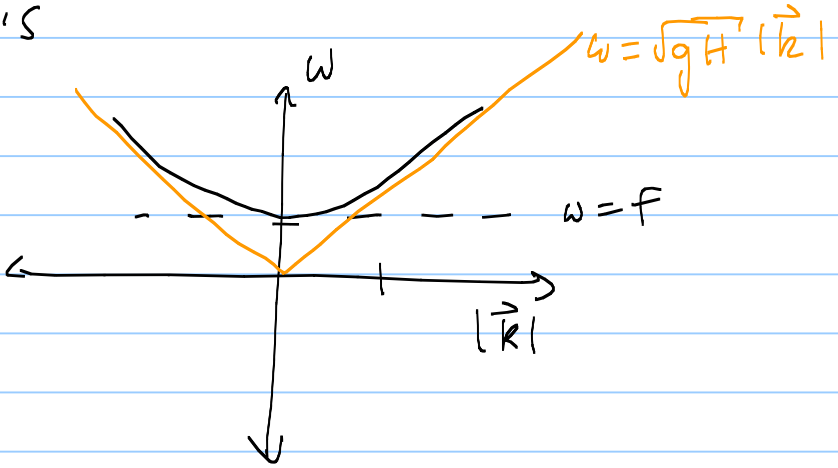
$\omega = 0$ → geostrophic flow

$$-fv = -g\frac{\partial h}{\partial x} \qquad fu = -g\frac{\partial h}{\partial y}$$

$$\boxed{\omega^2 = f^2 + gH(l^2 + k^2)}$$

Dispersion relation for Poincaré waves

Plotting this



There are two interesting limits

① $|\vec{k}|L_r \gg 1$

Short wave limit

Wavelength small compared to Rossby radius

② $|\vec{k}| L_r << 1$

long wave limit

Wavelength large compared to Rossby radius

Note that :

$$\frac{\omega^2}{f^2} = 1 + \frac{gH}{f^2} |\vec{k}|^2$$

$$= 1 + |\vec{k}|^2 L_r^2$$

So that in limit ① $|\vec{k}| L_r >> 1$

$$\omega^2 = f^2 L_r^2 |\vec{k}|^2$$

or

$$\omega = \sqrt{gH} \, |\vec{k}|$$

i.e. the waves are non-dispersive and travel at the shallow water wave speed. In this limit the waves have properties identical to waves in the non-rotating system.

In limit ② $|\vec{k}| L_r << 1$

$$\omega^2 = f^2$$

i.e. the motions are inertial oscillations.

This illustrates how the ratio of the length scale of the flow to the Rossby radius of deformation

$$\frac{L}{L_r}$$

is the critical parameter that determines whether or not the flow feels the effects of rotation.

Do Poincaré waves have a signal in the PV field?

Recall that the linearized PV is:

$$q = \frac{f}{H} + \underbrace{\frac{\zeta}{H} - \frac{\eta f}{H^2}}_{q'}$$

What is q' associated w/ these waves

$$q' = \frac{1}{H}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) - \eta\frac{f}{H^2}$$

$$\begin{pmatrix} u \\ v \\ \eta \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \\ \eta_0 \end{pmatrix} e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

$$q' = \frac{1}{H}\left(i v_0 k - i u_0 l\right) - \eta_0\frac{f}{H^2}$$

Using the polarization relations:

$$v_0 = \frac{(if\omega - gH\ell k)}{(gH\ell^2 - \omega^2)}\, u_0$$

$$\eta_0 = H\, \frac{(i\ell f - \omega k)}{(gH\ell^2 - \omega^2)}\, u_0$$

$$q' = \left\{ \frac{1}{H}\left[\frac{ik\,(if\omega - gH\ell k)}{(gH\ell^2 - \omega^2)} - i\ell \right] - \frac{f}{H}\,\frac{(i\ell f - \omega k)}{(gH\ell^2 - \omega^2)} \right\} * \; u_0\, e^{+i(\vec{k}\cdot\vec{x} - \omega t)}$$

The term in parentheses becomes:

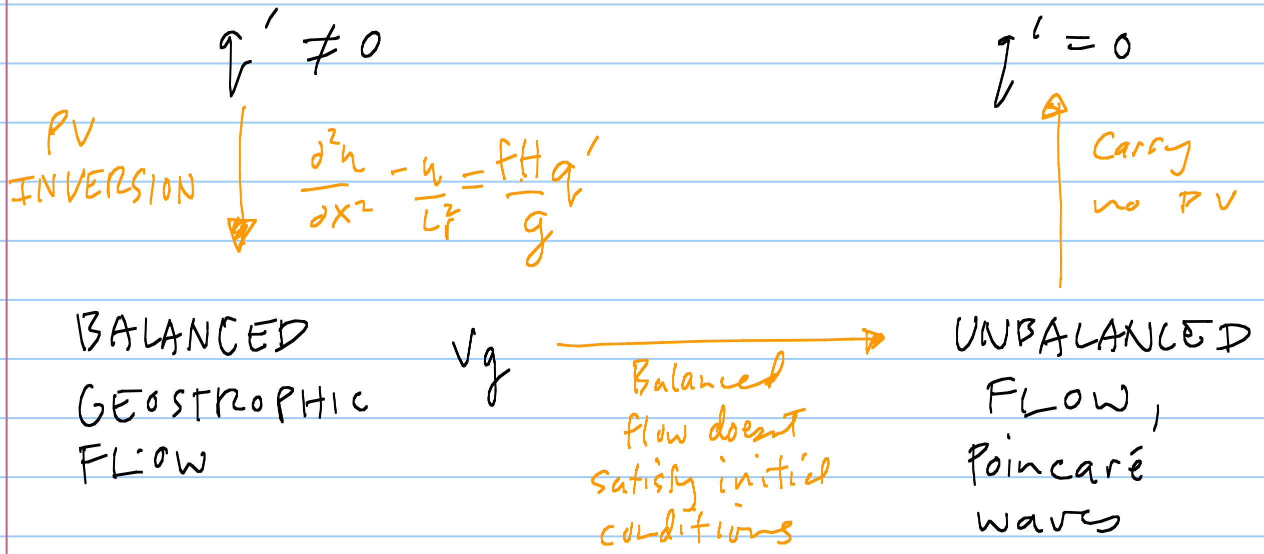
$$\frac{1}{H(gH\ell^2 - \omega^2)} \Big[ik(if\omega - gH\ell k) - i\ell(gH\ell^2 - \omega^2) - if\ell^2 + f\omega k \Big]$$

$$\frac{1}{H(gH\ell^2 - \omega^2)} \left\{ -k\cancel{f}\omega + f\cancel{k}\omega + i\ell\Big[\underbrace{\omega^2 - gH(k^2+\ell^2) - f^2}_{=0}\Big] \right\}$$

$$\Rightarrow \quad \boxed{q' = 0}$$

This is a very important result as it shows that!

the time dependent Poincaré waves CARRY NO POTENTIAL VORTICITY and thus are distinct from the geostrophic, balanced flow, which "carries" all the PV.

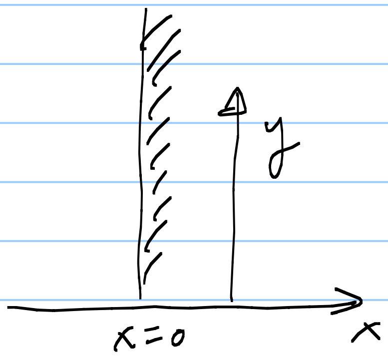


Another type of wavelike motion that carries no PV are Kelvin waves. Kelvin waves are a type of gravity wave that exists in the presence of rotation an a lateral boundary (or the Equator as you will discover in solving this week's homework).

Kelvin waves are generated during the gravitational adjustment of the fluid when lateral boundaries, i.e. coasts are present.

→ show animation of Kelvin waves generated during the 97–98 El Niño.

What are the properties of these waves and which direction do they propagate. This is easiest to show using the shallow water equations with a simple coastline running in the N-S direction.



At $x = 0$ $u = 0$ (i.e. there can be no flow through the boundary).

A Kelvin wave is a wave that only has flow in the direction of the coastline and no flow normal to it

$$v \neq 0 \qquad u = 0 \text{ everywhere}$$

With $u = 0$ everywhere the linearized shallow water equations become:

Ⓐ $$-fv = -g\frac{\partial \eta}{\partial x}$$ Flow geostrophic

Ⓑ $$\frac{\partial v}{\partial t} = -g\frac{\partial \eta}{\partial y}$$

Ⓒ $$\frac{\partial \eta}{\partial t} = -H\frac{\partial v}{\partial y}$$

Equations Ⓑ & Ⓒ can be combined into a single wave equation:

$$\frac{\partial^2 \eta}{\partial t^2} = -H\frac{\partial^2 v}{\partial t\,\partial y} = +gH\frac{\partial^2 \eta}{\partial y^2}$$

Lets assume a seperable form for the solution:

$$\eta(x,y,t) \sim G(x)\,\tilde{F}(y,t)$$

Substituting this form into the wave equation:

$$\frac{\partial^2 \tilde{F}}{\partial t^2} - c^2\frac{\partial^2 \tilde{F}}{\partial y^2} = 0 \qquad c = \sqrt{gH}$$

The general solution being:

$$\tilde{F} = F(y-ct) \qquad \text{or} \qquad \tilde{F} = F(y+ct)$$

waves traveling north — waves traveling south

What is the E-W structure of these two oppositly propagating waves?

From equations Ⓐ & Ⓑ we can eliminate v from the equations:

$$-f\frac{\partial v}{\partial t} = -g\frac{\partial^2 \eta}{\partial x\,\partial t}$$

$$\frac{\partial v}{\partial t} = -g\frac{\partial \eta}{\partial y}$$

$$f g \frac{\partial \eta}{\partial y} = -g \frac{\partial^2 \eta}{\partial x \partial t}$$

$$f \frac{\partial \eta}{\partial y} = - \frac{\partial^2 \eta}{\partial x \partial t}$$

Substituting a solution of the form:

$$\eta = G(x) F(y-ct)$$

$$f F'(y-ct) G(x) = + c F'(y-ct) \frac{dG}{dx}$$

$$\frac{dG}{dx} = \frac{f}{c} G = \frac{G}{L_r}$$

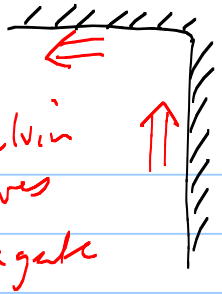
The solution of which is $G(x) = \exp[\mathrm{sgn}(f) x / L_r]$ so for the northward propagating wave the solution is:

$$① \quad \eta = F(y-ct) \exp\left[\mathrm{sgn}(f) \frac{x}{L_r}\right]$$

The solution for the southward propagating solution is

$$② \quad \eta = F(y+ct) \exp\left[-\mathrm{sgn}(f) \frac{x}{L_r}\right]$$

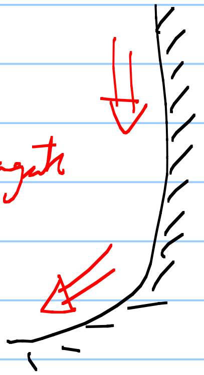
We require that as $x \to -\infty$ the solutions remain bounded, This means that in the northern (southern) hemisphere we must pick solution ① (②).



Kelvin waves propagate with the coast to their right

Northern hemisphere

$f > 0$



Kelvin waves propagate with coast to their left

Southern hemisphere

$f < 0$

Do Kelvin waves "carry" PV?

What is q' associated with Kelvin waves?

$$q' = \frac{\zeta}{H} - \frac{\eta f}{H^2}$$

$$= \frac{1}{H}\frac{\partial v}{\partial x} - \eta\frac{f}{H^2} = \frac{g}{fH}\underset{\uparrow}{\frac{\partial^2 \eta}{\partial x^2}} - \eta\frac{f}{H^2}$$

v associated w/ Kelvin wave is geostrophic

The solution for a Kelvin wave (in N. hemisphere)

$$\eta = F(y-ct)\exp\left(\frac{x}{L_r}\right)$$

So $$\frac{\partial^2 \eta}{\partial x^2} = F(y-ct)\exp\left(\frac{x}{L_r}\right)\frac{1}{(L_r)^2} = \eta\frac{f^2}{gH}$$

$$q' = \frac{g}{fH}\frac{\partial^2 \eta}{\partial x^2} - \eta \frac{f}{H^2}$$

$$= \frac{g}{fH}(\eta)\frac{f^2}{gH} - \eta\frac{f}{H^2}$$

$$\Rightarrow \quad \boxed{q' = 0}$$

In spite of the fact that the velocity of Kelvin waves is geostrophic, they carry no PV.

BALANCED
MOTIONS
CARRY PV

UNBALANCED
MOTIONS (Poincaré,
Kelvin waves) HAVE
NO PV SIGNATURE